SET THEORY HOMEWORK 3

Due Monday, March 20.

Problem 1. *Show that if κ is a regular uncountable cardinal in L, then L_κ satisfies all the axioms $ZF\setminus$ Powerset.*

Problem 2. *Suppose that $M \prec L_{\omega_1}$. Show that M is transitive. (Hint: for $X \in M$, take the $\prec_L$-least onto $f : \omega \to X$. Show that f is definable in L_{ω_1} from X and use this to show that $f \in M$. Also show $\omega \subset M$. Use these to prove that range of f is a subset of M)*

Problem 3. *Let κ be a regular uncountable cardinal.*

(1) *Suppose that $\tau < \kappa$ and $\langle C_i \mid i < \tau\rangle$ is a family of club subsets of κ. Show that $\bigcap_{i<\tau} C_i$ is also a club subset of κ.*
(2) *Suppose in addition that κ is inaccessible. Show that $\{\tau < \kappa \mid \tau \text{ is a cardinal }\}$ is club in κ. Show that $\{\tau < \kappa \mid \tau \text{ is a regular cardinal }\}$ is not a club.*

Problem 4. *Suppose that $\mathcal{F} \subset \mathcal{P}(\kappa)$ is a κ-complete normal ultrafilter on κ. I.e. $\mathcal{F}$ satisfies the following:*

- *for all $A \subset \kappa$, $A \in \mathcal{F}$ or $\kappa \setminus A \in \mathcal{F}$,*
- *if $\tau < \kappa$ and $\{A_\xi \mid \xi < \tau\}$ are sets in $\mathcal{F}$, then $\bigcap_{\xi<\tau} A_\xi \in \mathcal{F}$.*
- *for all regressive functions $f : \kappa \to \kappa$ (regressive means that $f(\alpha) < \alpha$ for all α), there is $\gamma < \kappa$ such that $f^{-1}(\gamma) := \{\alpha < \kappa \mid f(\alpha) = \gamma\}$ is in $\mathcal{F}$.*

Show that $\mathcal{F}$ is closed under diagonal intersections of length κ.